Spectral Basis Theory for the Identification of Structural Dynamic Systems

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This paper presents the theoretical formulation of a technique for determining the mathematical model of structural dynamic systems. This technique utilizes the fact that the forced responses of a structure can be decomposed into linear combinations of sets of frequency-dependent functions. The orthonormalized forms of these functions, along with the associated transformation matrices, are used to obtain the unique pseudoinverse of the matrix of forced responses, which satisfies all the required Moore-Penrose conditions. Variations of this scheme can be used to determine: 1) the damping matrix, when the mass and stiffness matrices are known; 2) the complex stiffness matrix, when the mass matrix is known; 3) the mass matrix, when the stiffness matrix is known; and 4) the reduced-order mass and stiffness matrices, when the larger-order matrices are known. Discussed are problems of redundancy and overtruncation of dynamic models and possible methods of avoiding them.

Nomenclature

$\boldsymbol{A}$	= transformation matrix defined in Eq. (7)
a	= modal coefficient
$\boldsymbol{a}$	= matrix defined in Eq. (16)
$\boldsymbol{C}$	= viscous damping matrix
$\boldsymbol{D}$	= hysteretic damping matrix
diag( )	= diagonal matrix
e_r, e_i	= random numbers
$\boldsymbol{F}$	= matrix of excitation spectra
F	= number of frequency measurement values
$\mathcal{F}_n(\omega)$	= function of frequency and modal parameters
$f(t)$	= time domain excitation vector
$f(\omega)$	= frequency domain excitation vector
f	= frequency index
$\boldsymbol{I}, I$	= identity matrices
i	= imaginary number
i, j	= indices
$\boldsymbol{K}$	= stiffness matrix
$\boldsymbol{K}_{cmplx}$	= complex stiffness matrix
L	= dimension of large-order model
$\boldsymbol{M}$	= mass matrix
N, N_1, N_2	= number of terms in modal series
R	= dimension of reduced order model
t	= time
$\boldsymbol{X}$	= matrix of response spectra
$\boldsymbol{x}(t)$	= time domain response vector
$\boldsymbol{x}(\omega)$	= frequency domain response vectors
$\boldsymbol{Y}(\omega)$	= matrix mobilities (or transfer functions)
$\boldsymbol{Y}$	= matrix defined in Eq. (A2)
$\boldsymbol{Z}$	= impedance matrix defined in Eq. (30)
λ	= eigenvalue
$\boldsymbol{\varphi}$	= spectral basis matrix defined in Eq. (7)
ψ	= eigenvector
$\boldsymbol{\Psi}$	= matrix of eigenvectors
ω	= frequency
$\boldsymbol{\Omega}$	= matrix defined in Eq. (A6)
$(\ )^*$	= conjugate matrix
$(\ )^T$	= transpose matrix
$(\ )^{-1}$	= inverse matrix
$(\ )^+$	= pseudoinverse matrix

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Introduction

THE identification of a physical system has the primary objective of deriving a mathematical model that reproduces the observed behavior of that system.[1-3] The differences in the approaches to and the techniques of system identification arise when consideration is given to the intended use of this mathematical model. In problems of control synthesis, it is sufficient to reproduce the response of the system to a given input, without a priori knowledge of the actual description of the physical system. The identification scheme may also need to be fast enough for on-line implementation, so that changes in the system characteristics during actual operation can be accounted for.

For structural dynamic systems, the identified mathematical model is needed, not only for realizing the responses of the structure to known excitations, but also for predicting the effects of design changes.[4,5] For these applications, it is essential for the mathematical model to be formulated in a manner that bears a physical relationship to the structure. In other words, the form of the mathematical model must be consistent with the equations governing the motions of the system. The discretized forms of these equations[6-8] are systems of second-order differential equations with constant coefficients, which constitute elements of the mass, stiffness, and damping matrices.

Simple modifications of the structure result in changes in mass, stiffness, and/or damping matrices. If, as it often happens, a model is derived that reproduces the forced response of a structure, but is not unique to that structure, the responses of the modified structure may not be well predicted by simple modifications of such a model.[4] This fact makes it imperative to establish the uniqueness of a structural dynamic model, in addition to reproducing the forced responses. The success or limitations of the various structural dynamic identification schemes that have been proposed to date can be understood and assessed in terms of the information used to assure the uniqueness of the mathematical model. In the context of this discussion, the term *uniqueness* refers to a model that, for the same order and the same measurement degrees of freedom, remains unchanged under different excitation conditions. It is well known that the elements of the mass, stiffness, or damping matrices do not remain the same when either the order of the model or the response coordinates are changed.[9]

Analytical techniques based on the finite-element method (e.g., NASTRAN[10]) require a detailed physical description of the structure. In principle, if the description is detailed enough, a unique large-order model of the undamped system is obtainable. The resulting order of the mathematical model is necessarily large because of the level of detail demanded by complex structures. In many instances, it may not be possible to include an adequate representation of the damping mechanisms in the analytical model,[11] thus necessitating some form of damping identification from experimental data. A variety of condensation schemes have been proposed for the reduction of the model order, many of which are derivatives of the reduction scheme proposed by Guyan,[12] more than 20 years ago. The drawback of this scheme for the dynamic problems is the poor accuracy of the reduced-order model over a broad range of frequencies.[13] Other approaches to dynamic condensation have been proposed using iterative procedures to match the eigenvalues of the large-order system with those of the reduced model, within the frequency range of interest.[14–16]

Experimental techniques based on modal testing seek to determine modal parameters such as natural frequencies, mode shapes, damping coefficients, and generalized masses (or generalized stiffnesses) from measurements of the structural mobilities (or transfer functions).[17,18] In recent years, these approaches have become widespread due to the availability of computing devices that facilitate the direct measurement of the frequency response functions. Modal identification schemes using data in the time domain have also been published.[19,20] If the structure is a discrete system with a finite number of degrees of freedom and if one succeeds in identifying as many modal parameters as there are degrees of freedom, Berman and Flannelly[4] have shown how to reconstruct the mass, stiffness, and damping matrices from this information. Indeed, many identification schemes based on this approach work fine when tested on made-up, discrete examples, but somehow seem to fall apart when applied to real structures with distributed parameters.[13] The *incomplete models* that are obtained when the number of modes available is less than the order of the model[4,21,22] generally turn out to be nonunique.

Recent approaches to structural systems identification have utilized information from both analysis and experiment.[23–27] These are often referred to as model *update* schemes. The purpose is to use analytical methods to obtain a first approximation to the system matrices and then to use modal test data in conjunction with a model adjustment algorithm to arrive at an improved model that shows agreement with the measured modal data. Even when this approach is taken, the issue of uniqueness of the final model is not completely resolved. Questions still remain about the integrity of the experimentally determined modes vis-a-vis the analytically derived modes.[13] One is naturally reluctant to change an accurate analytical model to fit inaccurate modal data and vice versa. Another way of combining analytical and experimental information is to use the forced response data directly. This approach often entails the pseudoinversion of rectangular matrices. Here also, there are questions of uniqueness of the identified model. There is a characteristic trap that many researchers fall into when attempting to extract system matrices from measured data *alone*. In their original paper, Berman and Flannelly[4] showed why the information about the impedance matrices for a particular structure may not be present in the measured responses alone. Unless the mathematical model is complete (i.e., the physical system is discrete and the order of the model includes all the degrees of freedom[28–30]), the derived matrices may not bear any relationship to the given structure. A detailed discussion of the nonuniqueness of models identified from measured data alone is presented in the Appendix.

In this paper, the theoretical formulation of an approach to system identification is presented. This method is applicable in situations where it has been possible to determine the stiffness and/or mass matrices that are unique to the structure. Such is the case, for example, when the large-order stiffness matrix has been statically condensed (a common practice in stress analysis), when the mass matrix can be determined by lumping (e.g., structures with concentrated inertial components), or when a dynamic model is sought based on a statically verified stiffness matrix (i.e., after a static test). This information, along with forced response measurements (mobilities or transfer functions), are then used to derive the remaining unknown matrices of the system. The contribution of this paper lies in the manner in which the issues of broadband accuracy and uniqueness are resolved. It turns out that the selection of the response coordinates and the order of the mathematical model play important roles in assuring both broadband accuracy and uniqueness of the identified model. It is shown that the main cause of nonuniqueness is redundancy of response coordinates (i.e., coordinates where the structural responses are linear combinations of responses elsewhere, an obvious example being the case of two accelerometers positioned so closely that they yield proportional measurements), while poor accuracy may be caused by overtruncation of the model (a fact pointed out in Ref. 4).

Theory

The discretized equations governing the vibration of linear structures are, for a viscously damped model,

$$M\frac{d^2x(t)}{dt^2}+C\frac{dx(t)}{dt}+Kx(t)=f(t) \tag{1}$$

and, for a hysteretically damped model,

$$M\frac{d^2x(t)}{dt^2}+(K+iD)x(t)=f(t) \tag{2}$$

where M, C, K, and D are the $R\times R$ mass, viscous damping, stiffness, and hysteretic damping matrices of the R degree of freedom model and $f(t)$ and $x(t)$ are the $R\times 1$ force and response vectors, respectively.

The forced vibratory response of systems described by Eq. (1) or (2) can be determined from Fourier transforms of these equations,

$$(-\omega^2 M+i\omega C+K)x(\omega)=f(\omega) \tag{3}$$

or

$$(-\omega^2 M+K+iD)x(\omega)=f(\omega) \tag{4}$$

where $x(\omega)$ and $f(\omega)$ are transforms of the response and forcing vectors, respectively. If the order of the truncated model is R, then $x(\omega)$ and $f(\omega)$ are $R\times 1$ vectors and M, C, K, and D are $R\times R$ matrices. If the forcing and response vectors are measured at F discrete frequencies, then Eqs. (3) and (4) can be written as

$$-MX\,\mathbf{diag}(\omega_f^2)+iCX\,\mathbf{diag}(\omega_f)+KX=F \tag{5}$$

or

$$-MX\,\mathbf{diag}(\omega_f^2)+(K+iD)X=F \tag{6}$$

where X and F are $R\times F$ matrices, the fth columns of which are the $x(\omega_f)$ and $f(\omega_f)$ vectors, respectively ($f=1,2,...,F$), and $\mathbf{diag}(\omega_f^2)$ and $\mathbf{diag}(\omega_f)$ are $F\times F$ diagonal matrices, the elements of which are ω_f^2 and ω_f, respectively ($f=1,2,...,F$). In general, $F\gg R$.

If the matrix of forced responses X can be expressed in the form,

$$X = A\varphi \tag{7}$$

where A is an $R \times R$ nonsingular transformation matrix and φ is an $R \times F$ orthonormal spectral basis matrix such that

$$\varphi\varphi^{*T} = I \tag{8}$$

with φ^{*T} being the complex conjugate transpose of φ and I the $R \times R$ identity matrix, then the unique right-hand pseudoinverse of the forced response matrix is

$$X^{+} = \varphi^{*T}A^{-1} \tag{9}$$

where A^{-1} is the inverse of the A matrix.

The pseudoinverse given by Eq. (9) satisfies the following four Moore-Penrose[31] conditions:

$$X^{+}XX^{+} = \varphi^{*T}A^{-1}A\varphi\varphi^{*T}A^{-1} = \varphi^{*T}A^{-1} = X^{+} \tag{10}$$

$$XX^{+}X = A\varphi\varphi^{*T}A^{-1}A\varphi = A\varphi = X \tag{11}$$

$$(XX^{+})^{*T} = (A\varphi\varphi^{*}A^{-1})^{*T} = (AA^{-1})^{*T} = I = XX^{+} \tag{12}$$

$$(X^{+}X)^{*T} = (\varphi^{*T}A^{-1}A\varphi)^{*T} = (\varphi^{*T}\varphi)^{*T} = \varphi^{*T}\varphi = X^{+}X \tag{13}$$

From the numerical standpoint, the transformation in Eq. (7) is always possible provided that the matrix X is of rank R. However, it is not sufficient that the numerical rank of this matrix be maximal, since measurement or round-off errors can disguise true deficiencies in its rank. If elements of the rows of the orthonormal spectral basis matrix are plotted against frequency, it is possible to determine how many of these rows have their origin in the true dynamic behavior of the structure.

To demonstrate the frequency characteristics of the rows of the orthonormal basis matrix φ, it is necessary to consider the relationship between the response and the excitation via a transfer function matrix,

$$x(\omega) = Y(\omega)f(\omega) \tag{14}$$

where $Y(\omega)$ is an $R \times R$ matrix of transfer functions, the ijth element of which can be written as a series[4,32] as

$$Y_{ij}(\omega) = \sum_{n=1}^{N} a_{in}a_{jn}\mathscr{F}_n(\omega) \tag{15a}$$

where a_{in} and a_{jn} relate to the nth modal coefficients at the ith and jth degrees of freedom, and $\mathscr{F}_n(\omega)$ is a function of the excitation frequency ω and the nth modal parameters such as natural frequency and damping coefficients,

$$\mathscr{F}_n(\omega) = \frac{1}{\Omega_n^2[1-(\omega^2/\Omega_n^2) + ig_n(\omega)]} \tag{15b}$$

where Ω_n, and g_n are the natural frequency and damping coefficient of the nth mode, respectively. In the following developments, it will be assumed that for system identification purposes, the applied forces $f(\omega)$ are independent of frequency. In other words, the forced responses are composed of linear combinations of the mobilities. This practical convenience turns out to simplify the mathematics quite a bit.

The number of terms in the series of Eq. (15) for a continuous (distributed parameter) system should, in principle, be infinite. However, within a selected frequency range, sufficient accuracy can be achieved with a finite number of terms. From a practical standpoint, in order for a truncated model to accurately reproduce the response of a structure over a selected frequency range, the number of terms in Eq. (15) should be no less than the number of structural resonances occurring in that frequency range; but it could be more, due to the effects of high- and low-frequency residual modes.

If a_n is an $R \times R$ matrix with

$$(a_n)_{ij} = a_{in}a_{jn} \tag{16}$$

and $\mathbf{diag}[\mathscr{F}_n(\omega_f)]$ is an $F \times F$ diagonal matrix of elements, $\mathscr{F}_n(\omega_f)$ $(f = 1,2,...,F)$, then the $R \times F$ forced response matrix can be written, using Eqs. (14) and (15), as

$$X = \sum_{n=1}^{N} a_n F\,\mathbf{diag}[\mathscr{F}_n(\omega_f)] \tag{17}$$

Equation (17) represents the forced response matrix as a linear combination of N matrices, each of which is associated with a frequency function $\mathscr{F}_n(\omega)$. Because the arrays $\mathscr{F}_n(\omega_f)$ $(f = 1,2,...,F)$ are linearly independent for different values of n, the rank of X as prescribed by the true dynamic behavior of the structure will depend on the number of significant terms in the series of Eq. (17). Conversely, if the rows of X are resolved into linear combinations of orthonormal spectral basis vectors [i.e., the rows of φ in Eq. (7)], which are linear combinations of the arrays $\mathscr{F}_n(\omega_f)$, then the number of such basis vectors can be used to determine the dynamical rank (as opposed to numerical rank) of the forced response matrix.

In practice, since the identification of the arrays $\mathscr{F}_n(\omega)$ is a relatively difficult process, it is easier to determine the orthonormal spectral basis vectors from the mobilities $Y_{ij}(\omega_f)$ using the Gram-Schmidt orthogonalization scheme.[3] The transformation matrix is then calculated as

$$A = X\varphi^{*T} \tag{18}$$

If the A matrix turns out to be singular, then the forced response matrix X is dynamically rank deficient.

Let N_1 denote the minimum number of terms in the series of Eq. (15) below which the accuracy of the model to the system is not assured and N_2 be an upper bound on the number of terms in the series beyond which the additional contributions to the structural response are negligible for the frequency range of interest. The requirement for the truncated model to be both unique and accurate is that

$$N_1 \le R < N_2 \tag{19}$$

If $R < N_1$, the truncated model will be unique, because the rank of the forced response matrix in Eq. (17) is at least R and the forms in Eqs. (7) and (8) are feasible. However, this model may not reproduce the structural response over the required range of frequencies, because the truncated model will have fewer resonances than the structure in this frequency range. If $R > N_2$, then the dynamical rank of the forced response matrix in Eq. (17) is less than R, meaning that the forms of Eqs. (7) and (8) will not be valid for the forced response matrix and any computed pseudoinverse of the forced response matrix will not be unique to the structure because it did not originate from the true dynamics of the structure.

In general, N_1 can be determined by observing the number of structural resonances occurring within the frequency range of interest. The value of N_2 is usually not known a priori. However, in the process of transforming the forced response matrix X into the form of Eq. (7), subject to Eq. (8), it is possible to identify the redundant response coordinates that can be eliminated to assure that the rank of the forced response matrix X is at least R.

By further examining Eq. (17), it should be noted that the forces needed for generating a maximally ranked response matrix are those that excite all the principal structural modes within the frequency range of interest. Moreover, if the structural mobilities (or transfer functions) are treated as the measured forced responses, then the entries into the rows of the force matrix corresponding to the excitation coordinates are unity for all frequency values ω_f ($f = 1,2,...,F$). Thus, nearly noise-free measurements of structural mobilities (or transfer functions), which are achievable using fast Fourier transform (FFT) analyzers,[33] can be used for the identification process.

System Identification

There are a number of practical situations where the theory presented here could improve the result of the identification process. It may appear at first glance that this is one more curve-fitting scheme using pseudoinversion. However, the most important contribution of this theory is embedded in Eqs. (7), (17), and (19). The process of constructing the matrices on the right-hand side of Eq. (7) not only uncovers the maximal dynamical rank of the forced response matrix, but it also determines any redundant degrees of freedom that would otherwise cause the value of R to violate the condition in Eq. (19).

The recommended procedure for applying this theory begins with the preprocessing of the measured forced responses with the objective of determining the spectral basis matrix φ. In the process of finding the rows of φ, redundant degrees of freedom will show up as linear combinations of previously determined rows. Assuming that the forced responses have been transformed into the form of Eq. (7) and that the constraints of Eq. (19) have been satisfied, the following expressions can be used to calculate the unknown matrices of the system, in terms of the forced responses and the known matrices. In these equations, $(\ )^*$, $(\ )^T$, and $(\ )^{*T}$ have been used to denote the complex conjugate, transpose, and complex conjugate transpose of the indicated matrices, respectively.

Damping Matrix

Using Eq. (5) for viscous damping

$$C = -i[F - KA\varphi + MA\varphi \ \mathrm{diag}(\omega_f^2)] \ \mathrm{diag}(1/\omega_f)\varphi^{*T}A^{-1} \tag{20}$$

and Eq. (6) for hysteretic damping,

$$D = -i\{[F + MA\varphi \ \mathrm{diag}(\omega_f^2)]\varphi^{*T}A^{-1} - K\} \tag{21}$$

It should be noted that only symmetric components of the real parts of the matrices calculated from Eqs. (20) and (21) can be attributed to the physical system. The antisymetric imaginary components do not dissipate any energy from the system. The way to extract the needed components is to calculate one-fourth of the sum of the calculated matrix, its complex conjugate, transpose, and complex conjugate transpose.

Complex Stiffness Matrix

$$K = [F + MA\varphi \ \mathrm{diag}(\omega_f^2)]\varphi^{*T}A^{-1} \tag{22}$$

Again, it is the symmetric component of the result of Eq. (22) that is required. The antisymmetric component does not contribute to the potential energy and dissipation function of the system. The symmetric component is obtained from half the sum of the calculated matrix and its transpose.

Mass Matrix

If a viscous damping model is used, then Eq. (5) gives

$$M = [iCA\varphi \ \mathrm{diag}(\omega_f) + KA\varphi - F] \ \mathrm{diag}(1/\omega_f^2)\varphi^{*T}A^{-1} \tag{23}$$

or, in the case of a hysteretically damped system, from Eq. (6),

$$M = [(K + iD)A\varphi - F] \ \mathrm{diag}(1/\omega_f^2)\varphi^{*T}A^{-1} \tag{24}$$

Here also, it is necessary to retain only the symmetric components of the results of Eqs. (23) and (24), since the antisymmetric components do not contribute to the kinetic energy of the physical system.

Dynamic Condensation

Suppose that a detailed finite-element analysis of a given structure has resulted in the large-order $L \times L$ mass and stiffness matrices M_L and K_L, respectively. As a result of the detailed physical representation of the structure involved in the analysis, one can presume that this large-order model is unique to the structure. The objective of dynamic condensation is to obtain a reduced order $R \times R$ mass and stiffness matrices M_R and K_R, where $R \ll L$. These matrices should not only be unique to the structure, but should also match the dynamic response of the structure at the reduced coordinates over a desired frequency range $\omega_1 \le \omega < \omega_F$.

Let the large-order system matrices be partitioned as

$$M_L = \begin{pmatrix} M_{RR} & M_{RS} \\ M_{SR} & M_{SS} \end{pmatrix} \tag{25}$$

and

$$K_L = \begin{pmatrix} K_{RR} & K_{SR} \\ K_{SR} & K_{SS} \end{pmatrix} \tag{26}$$

where $S = L - R$ and the subscripts RR, RS, SR, and SS indicate the dimensions of the respective submatrices. The top-left submatrices correspond to the coordinates of the reduced-order model.

If F_R denotes an $R \times 1$ vector, every element of which is unity, then an $R \times F$ forced response matrix X_R can be assembled such that its fth column X_{Rf} is given by

$$X_{Rf} = [Z_{RRf} - Z_{RSf}Z_{SSf}^{-1}Z_{SRf}]^{-1}F_R \tag{27}$$

where

$$Z_{RRf} = -\omega_f^2 M_{RR} + K_{RR} \tag{28a}$$

$$Z_{RSf} = -\omega_f^2 M_{RS} + K_{RS} \tag{28b}$$

$$Z_{SRf} = -\omega_f^2 M_{SR} + K_{SR} \tag{28c}$$

$$Z_{SSf} = -\omega_f^2 M_{SS} + K_{SS} \tag{28d}$$

in which $f = 1,2,..., F$. The next step is to perform the transformation of the forced response matrix into the form of Eq. (7), subject to Eq. (8). In the process of doing this, any redundant degree of freedom should be eliminated from the set of reduced degrees of freedom. If the order of the reduced system must be preserved, then a replacement response coordinate should be selected and the procedure restarted. Familiarity with the structural design of the system should minimize the guesswork when selecting the response coordinates most likely to yield independent responses. When this process is completed, we have

$$X_R = A_R\varphi_R \tag{29}$$

with

$$\varphi_R\varphi_R^{*T} = I \tag{30}$$

where A_R is an $R \times R$, nonsingular coefficient matrix and φ_R is the $R \times F$ spectral basis matrix.

The stiffness matrix of the reduced-order system is obtained using the standard static reduction scheme,[8]

$$K_R = K_{RR} - K_{RS}K_{SS}^{-1}K_{SR} \tag{31}$$

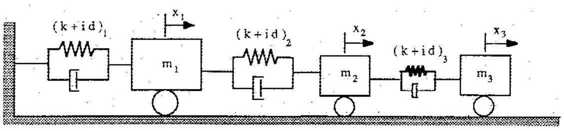

Fig. 1 Lumped spring-mass system with hysteretic damping.

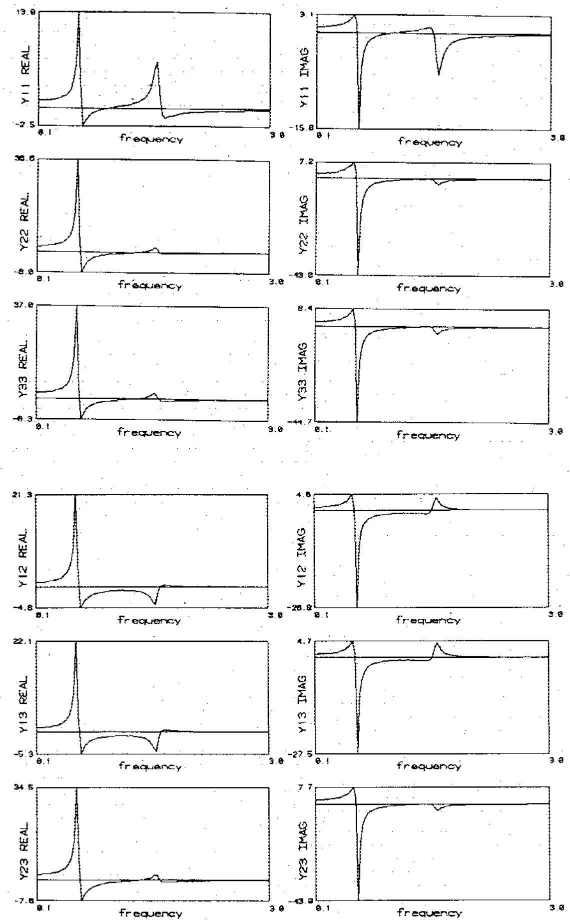

Fig. 2 Displacement mobilities with 5% random error.

If F_{RF} denotes an $R \times F$ matrix, all the elements of which are unity, then the mass matrix of the reduced-order system is given by the real, symmetric component of

$$M_R = [K_R A_R \varphi_R - F_{RF}] \,\mathrm{diag}(1/\omega_f^2) \varphi_R^{*T} A_R^{-1} \tag{32}$$

Numerical Example

As an illustration of the method of application of this theory, consider the arrangement of three lumped masses connected by hysteretically damped springs shown in Fig. 1: $m_1 = 1.0$ slugs, $m_2 = m_3 = 0.5$ slugs; $(k + id)_1 = (k + id)_2 = (1.0 + i0.05)$ lb/ft and $(k + id)_3 = (10.0 + i0.5)$ lb/ft. The frequency range of interest is between 0.1 and 3.0 rad/s. It is presumed that measurement errors of up to 5% of the instantaneous displacements will be present in the forced response data. A simulation of the measured displacement mobilities $Y_{ij}(\omega)$ ($i = 1,2,3; j = 1,2,3$) at any given frequency is accomplished by first solving for the response at i due to a unit force applied at j and then multiplying the result by a complex factor $[(1.0 + e_r) + i(1.0 + e_i)]$, where e_r and e_i are randomly generated numbers between -0.05 and $+0.05$. The plots of the six distinct mobilities Y_{11}, Y_{22}, Y_{33}, Y_{12}, Y_{13}, Y_{23} at 100 frequency points between 0.1 and 3.0 rad/s are shown in Fig. 2. It is seen that this frequency range includes only two of the system resonances.

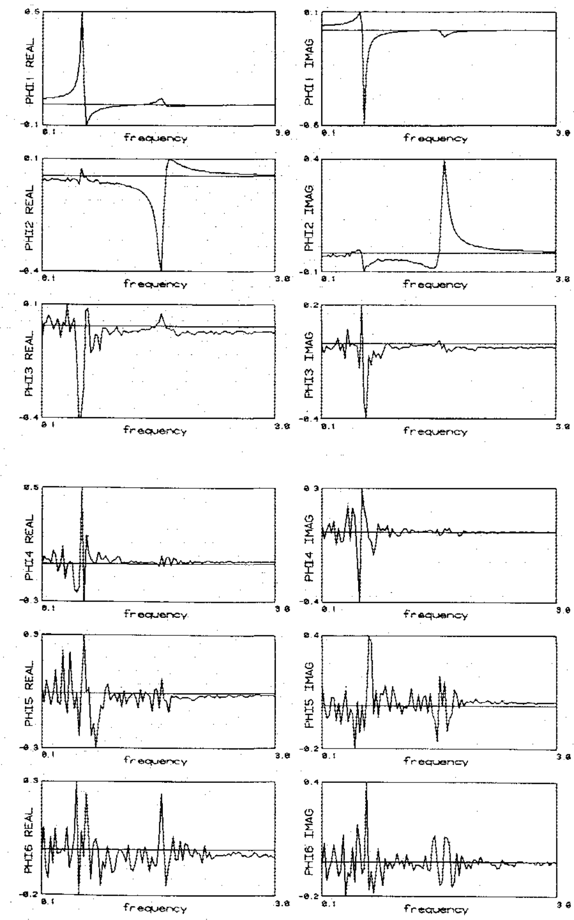

Fig. 3 Orthonormal spectral basis functions.

Using the Gram-Schmidt orthogonalization procedure, these six mobility arrays were then used to determine the orthonormal spectral basis arrays. Plots of these basis arrays are shown in Fig. 3. Of the six arrays plotted in this figure, only the first two exhibit the dynamical characteristics associated with the displacement mobilities. The remaining four basis arrays are obviously due to nondynamical information present in the mobilities. If any combination of forces (with constant frequency spectra) are used to excite this system within this frequency range, the maximum dynamical rank of the forced response matrix will be two. Because of the measurement noise, the numerical rank of the forced response matrix may be as high as the number of measurement locations. However, the theory presented here has shown that only a two-degree-of-freedom mathematical model will be both unique and accurate for describing this system within the frequency range specified. A one-degree-of-freedom model may be unique, but it would not be accurate, since it cannot model the second resonance occurring in this frequency range. A three-degree-of-freedom model may accurately reproduce the data used to determine it, but may not be unique to the structure, since the additional basis array needed to identify such a model originates not from the dynamical behavior of the structure, but from the random noise present in the data. A different set of data could yield another model just as accurate in reproducing the input data.

Conclusions

In structural dynamics system identification, the determination of a mathematical model that is both accurate and unique to the structure has traditionally been an elusive problem. A survey of current literature revealed that this problem has not yet been completely resolved.

This paper has presented some fairly simple procedures for eliminating the nonuniqueness that may be caused by redundancy of measurement degrees of freedom and/or the poor accuracy that may be due to overtruncation of the model.

It should be emphasized that for distributed parameter systems, a discrete mathematical model based on measured data alone may not have the required physical relationship with the structure. In order to avoid this problem, it is recommended that part of the unknown parameters of the system be derived from a physical description of the system. Thus, the methods presented here can be used to obtain reliable estimates of 1) the damping matrix when both the mass and stiffness matrices are known, 2) the complex stiffness matrix when the mass matrix is known, 3) the mass matrix when the stiffness and damping matrices are known, and 4) the mass and stiffness matrices of the reduced-order model, when the mass and stiffness matrices of the large-order model have been obtained from a detailed-element analysis.

Appendix: Nonuniqueness of Models Identified from Measured Data Alone

A number of recently published papers[28-30] have proposed the determination of the mass, stiffness, and damping matrices of dynamical systems from measured forced response data. The basic scheme involves a least squares solution of Eq. (5) for the $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ matrices.

Let Eq. (5) be rearranged as

$$(-\boldsymbol{M} \mid i\boldsymbol{C} \mid \boldsymbol{K})\begin{bmatrix} \boldsymbol{X}\,\mathbf{diag}(\omega_f^2) \\ \boldsymbol{X}\,\mathbf{diag}(\omega_f) \\ \boldsymbol{X} \end{bmatrix} = \boldsymbol{F} \tag{A1}$$

and let a $3R \times F$ matrix $\boldsymbol{Y}_{3R\times F}$ be defined as

$$\boldsymbol{Y}_{3R\times F} = \begin{bmatrix} \boldsymbol{X}\,\mathbf{diag}(\omega_f^2) \\ \boldsymbol{X}\,\mathbf{diag}(\omega_f) \\ \boldsymbol{X} \end{bmatrix} \tag{A2}$$

These proposed methods indicate that if $F \gg 3R$, then a right-hand pseudoinverse of this matrix $\boldsymbol{X}^+_{F\times 3R}$ can be computed, so that the system matrix can found as

$$(-\boldsymbol{M} \mid i\boldsymbol{C} \mid \boldsymbol{K}) = \boldsymbol{F}\boldsymbol{Y}^+_{F\times 3R} \tag{A3}$$

In this Appendix, it will be shown that, regardless of how many frequency data points are used, the pseudoinverse $\boldsymbol{Y}^+_{F\times 3R}$ will be nonunique and so will be the identification of Eq. (A3).

From Eq. (A2) let

$$\boldsymbol{Y}_{3R\times F} = \boldsymbol{X}_{3R\times 3F}\boldsymbol{\Omega}_{3F\times F} \tag{A4}$$

where

$$\boldsymbol{X}_{3R\times 3F} = \begin{pmatrix} \boldsymbol{X} & 0 & 0 \\ 0 & \boldsymbol{X} & 0 \\ 0 & 0 & \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A\varphi} & 0 & 0 \\ 0 & \boldsymbol{A\varphi} & 0 \\ 0 & 0 & \boldsymbol{A\varphi} \end{pmatrix} \tag{A5}$$

and

$$\boldsymbol{\Omega}_{3F\times F} = \begin{pmatrix} \mathbf{diag}(\omega_f^2) \\ \mathbf{diag}(\omega_f) \\ \boldsymbol{I}_F \end{pmatrix} \tag{A6}$$

with $\boldsymbol{I}_F$ being an $F \times F$ identity matrix.

In order for a unique right-hand pseudoinverse of the $\boldsymbol{Y}_{3R\times 3F}$ matrix to exist, unique right-hand pseudoinverses of both the $\boldsymbol{\Omega}_{3F\times F}$ and the $\boldsymbol{X}_{3R\times 3F}$ matrices must be determinable. It has already been shown in this paper that, under appropriate conditions, a unique right-hand pseudoinverse of the $\boldsymbol{X}_{3R\times 3F}$ matrix can be found as

$$\boldsymbol{X}^+_{3F\times 3R} = \begin{pmatrix} \boldsymbol{\varphi}^T\boldsymbol{A}^{-1} & 0 & 0 \\ 0 & \boldsymbol{\varphi}^T\boldsymbol{A}^{-1} & 0 \\ 0 & 0 & \boldsymbol{\varphi}^T\boldsymbol{A}^{-1} \end{pmatrix} \tag{A7}$$

However, because there are more rows than columns in the $\boldsymbol{\Omega}_{3F\times F}$ matrix, a unique right-hand pseudoinverse does not exist for this matrix. It is noteworthy that this will always be the case whenever an attempt is made to calculate more than one of the system matrices ($\boldsymbol{M}$, $\boldsymbol{C}$, or $\boldsymbol{K}$) from forced response data alone. If Eq. (6) is used to model the system, the $\boldsymbol{K}$ and $\boldsymbol{D}$ matrices can be treated either separately as real-valued matrices or together as a complex-valued $\boldsymbol{K} + i\boldsymbol{D}$ matrix.

When the right-hand pseudoinverse of $\boldsymbol{Y}_{3R\times F}$ is calculated directly as

$$\boldsymbol{Y}^+_{F\times 3R} = \boldsymbol{Y}^T_{F\times 3R}(\boldsymbol{Y}_{3R\times F}\boldsymbol{Y}^T_{F\times 3R})^{-1} \tag{A8}$$

it would appear that the only requirement is that the $\boldsymbol{Y}_{3R\times F}$ matrix should have a rank of $3R$.

If time domain data are used for the identification process and the system is indeed discrete (e.g., a spring/mass/damper structure), with responses measured at all the degrees of freedom, then the transients present in such data (displacement, velocity, and acceleration), at all R degrees of freedom, have the effect of increasing the rank of this matrix. Thus, numerical examples based on synthesized discrete models tend to give such good results that it is easy to conclude that the scheme is reliable. The fact is that changes in the excitations altering the transient response will usually result in changes in the identified model.

In frequency domain identification, the results based on matching the eigenvalues and eigenvectors of the identified system with those of the synthesized examples do not prove uniqueness either. Indeed, for the same excitation and response spectra [$\boldsymbol{F}$ and $\boldsymbol{X}$, per the notation of Eqs. (5) and (6)] of an undamped system and for any arbitrary, symmetric, positive definite mass or stiffness matrix it is possible to compute the commentary stiffness or mass matrix, respectively, that will reproduce the given data. Such mass/stiffness pairs will therefore produce the same eigenvalues and eigenvectors. Worse still, proportional combinations of such pairs of mass and stiffness matrices will yield the same eigenvalues and eigenvectors. In order to see this, consider the eigenvalue problem,

$$(-\lambda^2\boldsymbol{M}_i + \boldsymbol{K}_i)\boldsymbol{\psi} = 0 \tag{A9}$$

where $\boldsymbol{M}_i$ and $\boldsymbol{K}_i$ are $R \times R$ mass and stiffness matrices, respectively. Let $\mathbf{diag}(\lambda_r^2)$ be an $R \times R$ diagonal matrix containing the eigenvalues of Eq. (A9) and $\boldsymbol{\Psi}$ be an $R \times R$ matrix, the rth column of which is the rth eigenvector of Eq. (A9) ($r = 1,2,\ldots,R$). Then,

$$\boldsymbol{M}_i\boldsymbol{\Psi}\,\mathbf{diag}(\lambda_r^2) = \boldsymbol{K}_i\boldsymbol{\Psi} \tag{A10}$$

For any specified $\boldsymbol{K}_i$ along with the same modal data (excluding rigid-body modes),

$$\boldsymbol{M}_i = \boldsymbol{K}_i\boldsymbol{\Psi}\,\mathbf{diag}(1/\lambda_r^2)\boldsymbol{\Psi}^{-1} \tag{A11}$$

or, if the mass matrix $\boldsymbol{M}_i$ is specified along with the modal data,

$$\boldsymbol{K}_i = \boldsymbol{M}_i\boldsymbol{\Psi}\,\mathbf{diag}(\lambda_r^2)\boldsymbol{\Psi}^{-1} \tag{A12}$$

The observations presented in the Appendix conclude that the identification of mathematical models of structural systems from measured response data alone may not be reliable enough for applications where it is important to maintain a unique relationship between the model and the physical structure. Fortunately, in modern design practice, detailed descriptions of the physical system will usually permit the determination of either the unique mass or the unique stiffness matrices for a given system. It is recommended that as much of such unique information about the system as possible should be used in the identification process.

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